ON FORMS OF THE GENERAL SOLUTION OF THE SPATIAL PROBLEM OF THE THEORY OF ELASTICITY WITH THE AID OF HARMONIC FUNCTIONS

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Let us represent the solution of the equation of equilibrium in terms of displacements

$$\Delta \mathbf{u} + \frac{1}{1-2\nu} \nabla^2 \cdot \mathbf{u} = 0 \qquad (1)$$

where Δ is the Laplace operator, Δ is the Hamilton operator [1], $\nabla^2 = \nabla\nabla$ (the operators are multiplied dyadically), ν is Poisson's ratio, and the dot indicates scalar multiplication in the form

$$\mathbf{u} = \alpha \mathbf{R} + \beta (\nabla \mathbf{R}) \cdot \mathbf{r} + \gamma \mathbf{r} \cdot (\nabla \mathbf{R}) + \delta \mathbf{r} (\nabla \cdot \mathbf{R}) + \varepsilon r^2 (\nabla^2 \cdot \mathbf{R}) \qquad (2)$$

where α, β, γ, δ, ϵ are constants subject to determination, $\mathbf{R}$ is an arbitrary harmonic vector, $\mathbf{r}$ is the radius vector, $r^2 = \mathbf{r}\cdot\mathbf{r}$. In expression (2), the terms with $\mathbf{R}$ and $\mathbf{r}\cdot(\nabla \mathbf{R})$ are harmonic and the terms with $(\nabla \mathbf{R})\cdot\mathbf{r}$, $\mathbf{r}(\nabla\cdot\mathbf{R})$ and $r^2(\nabla^2\mathbf{R})$ are biharmonic functions. Substituting (2) into (1), we obtain the conditions which have to be satisfied by the constants α, β, γ, δ, ϵ:

$$\gamma + \delta + 2(3-4\nu)\varepsilon = 0, \quad \alpha + (3-4\nu)\beta + 2\gamma + 2(3-2\nu)\delta + 4(2-3\nu)\varepsilon = 0 \qquad (3)$$

Having determined α and γ from (3), we obtain the solution of equation (1) in the form

$$\mathbf{u} = [(4\nu-3)\beta + 4(1-\nu)(\varepsilon-\delta)]\mathbf{R} + \beta(\nabla\mathbf{R})\cdot\mathbf{r} + \\ + [2(4\nu-3)\varepsilon - \delta]\mathbf{r}\cdot(\nabla\mathbf{R}) + \delta\mathbf{r}(\nabla\cdot\mathbf{R}) + \varepsilon r^2(\nabla^2\cdot\mathbf{R}) \qquad (4)$$

where β, δ, ϵ are now arbitrary constants which can be used to obtain all the so far known forms of the general solution, as well as a number of new ones.

Thus for $\delta = \epsilon = 0$, $\beta = -1$ we obtain the Popkovich-Neuber [2,3] solution

$$\mathbf{u}_1 = (3 - 4\nu)\,\mathbf{R}_1 - (\nabla\mathbf{R}_1)\cdot\mathbf{r} \tag{5}$$

For $\epsilon = \beta = 0$, $\delta = -1$ we obtain a solution indicated in papers [4-6]:

$$\mathbf{u}_2 = 4(1-\nu)\,\mathbf{R}_2 + \mathbf{r}\cdot(\nabla\mathbf{R}_2) - \mathbf{r}\,(\nabla\cdot\mathbf{R}_2) \tag{6}$$

For $\epsilon = -1$, $\delta = 2(1 - 2\nu)$, $\beta = -4(1-\nu)$ we obtain a solution indicated in [3]:

$$\mathbf{u}_3 = 2(1-2\nu)\,\mathbf{r}\,(\nabla\cdot\mathbf{R}_3) + 4(1-\nu)[\mathbf{r}\cdot(\nabla\mathbf{R}_3) - (\nabla\mathbf{R}_3)\cdot\mathbf{r}] - r^2\,(\nabla^2\cdot\mathbf{R}_3) \tag{7}$$

In addition to solutions (5) to (7) we obtain the following also from (4):

for

$$\beta = 0, \qquad \varepsilon = \frac{1}{7-8\nu}, \qquad \delta = \frac{2(4\nu-3)}{7-8\nu}$$

we have

$$\mathbf{u}_4 = 4(1-\nu)\,\mathbf{R}_4 + \frac{2(4\nu-3)}{7-8\nu}\,\mathbf{r}\,(\nabla\cdot\mathbf{R}_4) + \frac{1}{7-8\nu}\,r^2\,(\nabla^2\cdot\mathbf{R}_4) \tag{8}$$

for

$$\varepsilon = 1, \qquad \delta = 2(4\nu-3), \qquad \beta = \frac{4(7-8\nu)(1-\nu)}{3-4\nu}$$

we have

$$\mathbf{u}_5 = \frac{4(7-8\nu)(1-\nu)}{3-4\nu}\,(\nabla\mathbf{R}_5)\cdot\mathbf{r} + 2(4\nu-3)\,\mathbf{r}\,(\nabla\cdot\mathbf{R}_5) + r^2\,(\nabla^2\cdot\mathbf{R}_5) \tag{9}$$

for

$$\beta = 0, \qquad \varepsilon = \delta = \frac{1}{8\nu-7}$$

we have

$$\mathbf{u}_6 = \mathbf{r}\cdot(\nabla\mathbf{R}_6) + \frac{1}{8\nu-7}\,[\mathbf{r}\,(\nabla\cdot\mathbf{R}_6) + r^2\,(\nabla^2\cdot\mathbf{R}_6)] \tag{10}$$

for

$$\varepsilon = 0, \qquad \delta = -1, \quad \beta = 1$$

we have

$$\mathbf{u}_7 = \mathbf{R}_7 + (\nabla\mathbf{R}_7)\cdot\mathbf{r} + \mathbf{r}\cdot(\nabla\mathbf{R}_7) - \mathbf{r}\,(\nabla\cdot\mathbf{R}_7) \tag{11}$$

for

$$\varepsilon = 1, \qquad \delta = -8(1-\nu), \qquad \beta = 9 - 8\nu$$

we have

$$\mathbf{u}_8 = (9-8\nu)[\mathbf{R}_8 + (\nabla \mathbf{R}_8)\cdot\mathbf{r}] + 2\mathbf{r}\cdot(\nabla \mathbf{R}_8) - 8(1-\nu)\,\mathbf{r}(\nabla\cdot\mathbf{R}_8) + r^2(\nabla^2\cdot\mathbf{R} \qquad 12)$$

The solutions (8) to (12) are new. Solution (8) differs advantageously from the Papkovich-Neuber solution by the fact that it does not contain the gradient of the vector **R**, and as a consequence, is simpler in its structure than the latter (in solving a three-dimensional problem in curvilinear coordinates, it is simpler to find $\nabla\cdot\mathbf{R}$, than $\nabla\mathbf{R}$, and $\nabla\cdot\mathbf{R}$ may be found directly). In solution (9) all the terms are biharmonic functions. Solution (10) is rather simple. In addition to solutions (8) to (12), there follows from (4) an infinite number of five-term solutions of the type (12), though with other coefficients. Solutions (8) to (12) will be called general, because for them $\nabla\cdot\mathbf{u}\neq 0$, $\nabla\times\mathbf{u}\neq 0$ (an exception is represented by solution (11) for which $\nabla\cdot\mathbf{u}_7 = 0$, $\nabla\times\mathbf{u}_7\neq 0$). In fact, substituting β, δ, ϵ, corresponding to these solutions, into the expressions for the divergence and the rotation of the displacement vector

$$\nabla\times\mathbf{u} = -2(1-2\nu)[(\beta+\varepsilon+\delta)\nabla\cdot\mathbf{R} + 2\varepsilon\mathbf{r}\cdot(\nabla^2\cdot\mathbf{R})] \qquad (13)$$

$$\nabla\times\mathbf{u} = -[4(1-\nu)\beta + 2(1-2\nu)\varepsilon + (5-4\nu)\delta]\,\nabla\times\mathbf{R} + [2(4\nu-3)\varepsilon - \delta]\,\mathbf{r}\cdot(\nabla^2\times\mathbf{R}) + (\delta-2\varepsilon)(\nabla^2\cdot\mathbf{R})\times\mathbf{r} \qquad (14)$$

we see that the inequalities (13) are satisfied. A certain variety of forms of the solution may be obtained also on the basis that the algebraic sum of any solutions of equation (1) is also a solution of equation (1). Thus, setting $\mathbf{R}_6 = -(8\nu-7)\,\mathbf{R}_5$ we obtain a new solution in the form

$$u = \frac{u_5 - u_6}{8\nu-7} = -\frac{4(1-\nu)}{3-4\nu}(\nabla\mathbf{R})\cdot\mathbf{r} + \mathbf{r}(\nabla\cdot\mathbf{R}) - \mathbf{r}\cdot(\nabla\mathbf{R}) \qquad (15)$$

It is easy to show that in order to reduce the solution (4) to the form of the Papkovich-Neuber solution (5), the following conditions must be satisfied

$$\delta = -2(1-2\nu)\,\varepsilon, \qquad (7-8\nu)\,\delta + [2(3-4\nu)^2 - 4(1-\nu)]\,\varepsilon = 0 \qquad (16)$$

If these conditions are satisfied, solution (4) may be represented in the form

$$\mathbf{u} = (3-4\nu)\,\mathbf{D} - (\nabla\mathbf{D})\cdot\mathbf{r} \qquad (17)$$

where **D** is a harmonic factor having the form

$$\mathbf{D} = [-\beta + \delta - 2(4\nu-3)\varepsilon]\,\mathbf{R} + \frac{2(4\nu-3)\varepsilon - \delta}{4(1-\nu)}[\mathbf{r}\cdot(\nabla\mathbf{R}) - (\nabla\mathbf{R})\cdot\mathbf{r}] - \varepsilon\mathbf{r}(\nabla\cdot\mathbf{R}) \qquad (18)$$

From solutions (6) to (12) only solution (7) may be reduced to the form (17), because its coefficients β, δ, ϵ satisfy conditions (16). It should also be noted that the solutions (5) to (12) may be supplemented by a particular solution of equation (1) (such that $\nabla\cdot\mathbf{u}_r = 0$) in the form of $\mathbf{u} = \nabla F + \nabla \times \mathbf{T} + \mathbf{r} \cdot \nabla^3 \cdot \mathbf{S}$, where **F** is an arbitrary harmonic scalar function, **T** and **S** are arbitrary harmonic vectors. As was shown in [5,7] there sometimes occurs the necessity to retain in (19) either **F** or **T** and **S**. In conclusion we write down the expression for the stress tensor σ, corresponding to the representation of u in the form (4)

$$\sigma = \frac{E}{2(1+\nu)}\{[(4\nu-2)(\varepsilon+\beta)-(5-4\nu)\delta](\nabla\mathbf{R}+\mathbf{R}\nabla)-8\nu\varepsilon\mathbf{r}\cdot(\nabla^2\cdot\mathbf{R})\mathrm{I}+ \\ +[2(4\nu-3)\varepsilon-\delta][\mathbf{r}\cdot(\nabla^2\mathbf{R})+(\mathbf{R}\nabla^2)\cdot\mathbf{r}]+(\delta+2\varepsilon)[(\nabla^2\cdot\mathbf{R})\mathbf{r}+ \\ +\mathbf{r}(\nabla^2\cdot\mathbf{R})]+2\beta(\nabla^2\mathbf{R})\cdot\mathbf{r}+[2\delta(1-2\nu)-4\nu(\varepsilon+\beta)](\nabla\cdot\mathbf{R})\mathrm{I}+2\varepsilon r^2(\nabla^3\cdot\mathbf{R})\}$$

where E is the modulus of elasticity and I is the unit tensor.

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